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Fractional superLie algebras and groups

H Ahmedov[1], **A Yildiz**[1] **and Y Ucan**[2]

[1] Feza Gürsey Institute, PO Box 6, 81220, Çengelköy, Istanbul, Turkey
[2] Yildiz Technical University, Department of Mathematics, Besiktas, Istanbul, Turkey

E-mail: hagi@gursey.gov.tr

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Abstract
The nth root of a Lie algebra and its dual (that is the fractional supergroup) based on the permutation group S_n invariant forms is formulated in the Hopf algebra formalism. Detailed discussion of S_3-graded $sl(2)$ algebras is performed.

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1. Introduction

To arrive at a superalgebra one adds new elements Q_α to generators X_j of the corresponding Lie algebra and defines the relations

$$\{Q_\alpha, Q_\beta\} = b^j_{\alpha\beta} X_j. \tag{1}$$

Observing that the anticommutator in the above relation is invariant under the cyclic Z_2 or permutation S_2 groups we can look for possible generalization of the supersymmetry by using S_n or Z_n invariant structures instead of the anticommutator. For example, if $n = 3$ instead of (1) one has the cubic relation

$$Q_\alpha Q_\beta Q_\gamma + Q_\gamma Q_\alpha Q_\beta + Q_\beta Q_\gamma Q_\alpha = b^j_{\alpha\beta\gamma} X_j \tag{2}$$

which is Z_3 invariant and the cubic relation

$$Q_\alpha \{Q_\beta, Q_\gamma\} + Q_\beta \{Q_\alpha, Q_\gamma\} + Q_\gamma \{Q_\alpha, Q_\beta\} = b^j_{\alpha\beta\gamma} X_j \tag{3}$$

which is S_3 invariant. From the above relations only (3) appears to be consistent at the co-algebra level. Sometimes we will use the term fractional superalgebras for S_n-graded algebras with $n = 3, 4, \ldots$ with fractional supergroups being their dual.

Fractional superalgebras based on S_n invariant forms were first introduced in [1, 2]. In this paper we put this construction in the Hopf algebra context and define their dual, that is fractional supergroups. There are many reasons for doing that. In the formulation of superalgebras one can use either geometric (see e.g. [3]) or algebraic [4] approaches (see also [5] for comparison). As for fractional superalgebras, the geometric approach seems to be insufficient. This situation is similar to the theory of quantum algebras, where we have to work with universal enveloping algebras rather than with Lie algebras [6]. Moreover, having a fractional superalgebra in hand we can define fractional supergroups by taking the dual of the former. Having, at last, put fractional superalgebras in the Hopf algebra context we can use the well-developed representation theory of the latter in the construction of representations of fractional superalgebras.

There are other approaches to fractional supersymmetry in the literature [7–12]. For example, one can arrive at fractional supergroups by using quantum groups at the roots of unity [13].

The plan of the paper is as follows. To make the treatment reasonably self-consistent, in section 2 we give a formulation of superalgebras and groups in the Hopf algebra formalism. In section 3 we define fractional superalgebras and discuss the structure of their dual (fractional supergroups). Section 4 is devoted to the detailed discussion of the S_3-graded $sl(2)$ algebras.

2. Preliminaries on superalgebras

Let $U(g)$ be the universal enveloping algebra of a Lie algebra g generated by X_j, $j = 1, \ldots, \dim(g)$ with

$$[X_i, X_j] = \sum_{k=1}^{\dim(g)} c_{ij}^k X_k \tag{4}$$

where c_{ij}^k are the structure constants of the Lie algebra g. The Hopf algebra structure of $U(g)$ is given by the co-multiplication $\Delta : U(g) \rightarrow U(g) \otimes U(g)$, co-unit $\varepsilon : U(g) \rightarrow C$ and antipode $S : U(g) \rightarrow U(g)$:

$$\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j \qquad \varepsilon(X_j) = 0 \qquad S(X_j) = -X_j. \tag{5}$$

We can extend the Hopf algebra $U(g)$ by adding elements Q_α, $\alpha = 1, \ldots, N$ and K with relations

$$\{Q_\alpha, Q_\beta\} = \sum_{j=1}^{\dim(g)} b_{\alpha\beta}^j X_j \tag{6}$$

$$[Q_\alpha, X_j] = \sum_{\beta=1}^{N} a_{\alpha\beta}^j Q_\beta \tag{7}$$

$$K Q_\alpha = -Q_\alpha K \qquad K^2 = 1 \tag{8}$$

where $b_{\alpha\beta}^j$ and $a_{\alpha\beta}^j$ are the structure coefficients satisfying the superJacobi identities. This algebra which we denote by $U_2^N(g)$ can also be equipped with a Hopf algebra structure by defining

$$\Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha \qquad \Delta(K) = K \otimes K \tag{9}$$

$$\varepsilon(Q_j) = 0 \qquad \varepsilon(K) = 1 \qquad S(Q_j) = Q_j K \qquad S(K) = K. \tag{10}$$

The dual of $U_2^N(g)$ is the Hopf algebra $A_2^N(G) = C^\infty(G) \times \Lambda_2^N$, where $C^\infty(G)$ is the algebra of infinite differentiable functions on a Lie group G and Λ_2^N is the algebra over the field of complex numbers generated by θ_α, $j = 1, \ldots, N$ and λ with relations

$$\{\theta_\alpha, \theta_\beta\} = 0 \qquad \{\lambda, \theta_\alpha\} = 0 \qquad \lambda^2 = 1. \tag{11}$$

The operations Δ, ε and S in $A^N(G)$ depend on the value of the structure constants c_{ij}^k, $b_{\alpha\beta}^j$ and $a_{\alpha\beta}^j$.

For example, if $= C^N$ then the formulae

$$\Delta(\theta_\alpha) = \theta_\alpha \otimes 1 + \lambda \otimes \theta_\alpha \qquad \Delta(\lambda) = \lambda \otimes \lambda$$
$$\Delta(z_\alpha) = z_\alpha \otimes 1 + 1 \otimes z_\alpha + \lambda\theta_\alpha \otimes \theta_\alpha \tag{12}$$

$$\varepsilon(\theta_\alpha) = 0 \qquad \varepsilon(\lambda) = 1 \qquad \varepsilon(z_\alpha) = 0 \tag{13}$$

$$S(\theta_\alpha) = -\lambda\theta_\alpha \qquad S(\lambda) = \lambda \qquad S(z_\alpha) = -z_\alpha \tag{14}$$

define the super-N-dimensional translation group. The corresponding superalgebra is defined by

$$\{Q_\beta, Q_\alpha\} = \delta_{\beta\alpha} P_\alpha \qquad [X_\beta, X_\alpha] = 0 \qquad [Q_\beta, X_\alpha] = 0. \tag{15}$$

3. Fractional superalgebras and supergroups

To arrive at cubic root of a lie algebra g we have to replace the S_2 invariant form in (6) by the S_3 invariant one. Consequently, we define an algebra generated by X_j, $j = 1, \ldots, \dim(g)$ and $Q_\alpha, K, \alpha = 1, \ldots, N$ satisfying the relations (4) and

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = b^j_{\alpha\beta\gamma} X_j \tag{16}$$

$$[Q_\alpha, X_j] = a^j_{\alpha\beta} Q_\beta \tag{17}$$

and

$$K Q_\alpha = q Q_\alpha K \quad q^3 = 1 \qquad K^3 = 1 \tag{18}$$

where

$$\{Q_\alpha, Q_\beta, Q_\gamma\} \equiv Q_\alpha\{Q_\beta, Q_\gamma\} + Q_\beta\{Q_\alpha, Q_\gamma\} + Q_\gamma\{Q_\alpha, Q_\beta\} \tag{19}$$

is the S_3 invariant form. We denote this algebra by the symbol $U^N_3(g)$ with the lower index indicating the degree of grading. One can check that the above algebra is compatible with the co-algebra structure and antipode given by the formulae

$$\Delta(Q_\alpha) = Q_\alpha \otimes 1 + K \otimes Q_\alpha \qquad \Delta(K) = K \otimes K \tag{20}$$

$$\varepsilon(Q_j) = 0 \qquad \varepsilon(K) = 1 \qquad S(Q_j) = -K^2 Q_j \qquad S(K) = K^2. \tag{21}$$

For example, let us verify the consistence of the comultiplication Δ and (16). Since Δ is a homomorphism we have

$$\begin{aligned}\Delta(Q_\alpha Q_\beta Q_\gamma) = {} & Q_\alpha Q_\beta Q_\gamma \otimes 1 + 1 \otimes Q_\alpha Q_\beta Q_\gamma + Q_\alpha Q_\beta K \otimes Q_\gamma \\ & + Q_\alpha K Q_\gamma \otimes Q_\beta + K Q_\beta Q_\gamma \otimes Q_\alpha + Q_\alpha K^2 \otimes Q_\beta Q_\gamma \\ & + K Q_\beta K \otimes Q_\alpha Q_\gamma + K^2 Q_\gamma \otimes Q_\alpha Q_\beta.\end{aligned} \tag{22}$$

Using (18) we get

$$\sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha Q_\beta K \otimes Q_\gamma + Q_\alpha K Q_\gamma \otimes Q_\beta + K Q_\beta Q_\gamma \otimes Q_\alpha) = 0 \tag{23}$$

and

$$\sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha K^2 \otimes Q_\beta Q_\gamma + K Q_\beta K \otimes Q_\alpha Q_\gamma + K^2 Q_\gamma \otimes Q_\alpha Q_\beta) = 0. \tag{24}$$

Thus we have shown that

$$\sum_{(\alpha\beta\gamma)\in S_3} \Delta(Q_\alpha Q_\beta Q_\gamma) = \sum_{(\alpha\beta\gamma)\in S_3} (Q_\alpha Q_\beta Q_\gamma \otimes 1 + 1 \otimes Q_\alpha Q_\beta Q_\gamma) \tag{25}$$

which together with the comultiplication rule (5) for the generators X_j implies the consistence of the comultiplication (20) and the relation (16).

To define structure constants $b^j_{\alpha\beta\gamma}$ and $a^j_{\alpha\beta}$ we have to derive identities involving the commutator and S_3 invariant form. One can check that the relations

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0 \tag{26}$$

$$[A,\{B,C,D\}] + \{[B,A],C,D\} + \{B,[C,A],D\} + \{B,C,[D,A]\} = 0 \tag{27}$$

and

$$[A,\{B,C,D\}] + [B,\{A,C,D\}] + [C,\{B,A,D\}] + [D,\{B,C,A\}] = 0 \tag{28}$$

are satisfied identically. For example, we verify the identity (27). Let $B = A_1$, $C = A_2$ and $D = A_3$. Then

$$[A,\{A_1,A_2,A_3\}] = \sum_{ijk\in S_3} ([A,A_i]A_jA_k + A_iA_j[A,A_k] + A_i[A,A_j]A_k). \tag{29}$$

Combining terms (123), (132) from the first sum in the right-hand side of the above equality, (231), (321) from the second sum and (213), (312) from the third sum we get $\{[A,A_1],A_2,A_3\}$. In a similar fashion we obtain $\{[A,A_2],A_1,A_3\}$ and $\{[A,A_3],A_2,A_1\}$. Thus

$$[A,\{A_1,A_2,A_3\}] = \{[A,A_1],A_2,A_3\} + \{[A,A_2],A_1,A_3\} + \{[A,A_3],A_2,A_1\} \tag{30}$$

which is the identity (27).

The one given by (26) is the usual Jacobi identity. Inserting

$$A = X_i \qquad B = X_j \qquad C = Q_\alpha \tag{31}$$

into (26) and using (17) and (4) we get

$$\sum_{\sigma=1}^{N} (a^i_{\alpha\sigma}a^j_{\sigma\beta} - a^j_{\alpha\sigma}a^i_{\sigma\beta}) = \sum_{k=1}^{\dim(g)} c^k_{ij}a^k_{\alpha\beta}. \tag{32}$$

Comparing the above relation with (4) we conclude that the $N \times N$ matrices $a^j \equiv (a^j_{\alpha\beta})^N_{\alpha,\beta=1}$ define a N-dimensional representation of a given Lie algebra. There are different possibilities in the choice of this representation. For example, if $g = sl(2)$ and $N = 2$ we can either use the scalar representation $a^j_{\alpha\beta} = 0$ or the spinor one in which $a^j_{\alpha\beta}$ are the Pauli matrices. Consequently, for fixed Lie algebra g and N we can define different superfractional algebras. To be more precise one has to add an additional index in the notation $U^N_3(g)$ which reflects the transformation law of supergenerators Q_α with respect to a given Lie algebra g. However, for the sake of simplicity we will not do this. Detailed discussion of this nonuniqueness is performed in the next section where we consider fractional superalgebras $sl(2)$.

Let us now consider restrictions on structure coefficients coming from the other identities. Inserting

$$A = X_k \qquad B = Q_\alpha \qquad C = Q_\beta \qquad D = Q_\gamma \tag{33}$$

into the identity (27) and

$$A = Q_\sigma \qquad B = Q_\alpha \qquad C = Q_\beta \qquad D = Q_\gamma \tag{34}$$

into (28) and using (16), (17) we arrive at the relations

$$\sum_{\sigma=1}^{N} (a^k_{\alpha\sigma}b^i_{\sigma\beta\gamma} + a^k_{\beta\sigma}b^i_{\sigma\alpha\gamma} + a^k_{\gamma\sigma}b^i_{\sigma\beta\alpha}) = \sum_{j=1}^{\dim g} c^i_{jk}b^j_{\alpha\beta\gamma} \tag{35}$$

and

$$\sum_{k=1}^{\dim g} (b^k_{\alpha\beta\gamma}a^k_{\sigma\tau} + b^k_{\sigma\alpha\beta}a^k_{\gamma\tau} + b^k_{\gamma\sigma\alpha}a^k_{\beta\tau} + b^k_{\beta\gamma\sigma}a^k_{\alpha\tau}) = 0. \tag{36}$$

Now we define fractional supergroups. Let $x = \{x_{nm}\}$ be the matrix representing a Lie group G and $A(G)$ be the algebra of polynomials on G. It is known that $A(G)$ is the Hopf algebra which is in nondegenerate duality with the universal enveloping algebra $U(g)$ [14]. In general the number of group elements x_{nm} is more than the number of generators X_j in the corresponding Lie algebra g. This is due to the fact that there may be some restrictions on the matrix representing a Lie group. For example, if $G = SL(2)$ we have a two by two matrix with determinant equal to 1. The number of independent group parameters is equal to the number of generators of $sl(2)$. Explicitly one can define these parameters by using some decomposition (Gauss, Cartan, Iwasawa and so on). In a similar way for an arbitrary matrix Lie group we can resolve restrictions imposed on the elements x_{nm} and obtain independent group parameters x_j with duality relations

$$\langle x_i, X_j \rangle = \delta_{ij} \tag{37}$$

where X_j are the generators of the corresponding Lie algebra. However, in general $A(G)$ in terms of these new parameters will not be the polynomial algebra. It appears that in the Hopf algebra formalism it is more convenient to work with elements x_{nm}. Instead of solving restrictions imposed on these element one defines new generators X_{nm} with some restrictions. For example, if $g = sl(2)$ we define four generators with the restriction $X_{11} + X_{22} = 0$.

To construct the dual algebra to a fractional superalgebra $U_3^N(g)$ we have to introduce new parameters θ_α, $\alpha = 1, \ldots, N$ and λ corresponding to the fractional supergenerators Q_α and K. The duality relations are given by the following formulae:

$$\langle \theta_\alpha, Q_\beta \rangle = \delta_{\alpha\beta} \qquad \langle \lambda, K \rangle = q \qquad \langle x_{nm}, K \rangle = \delta_{nm} \tag{38}$$

with all other linear relations being zero. Recall the property of the duality relations [14]

$$\langle ab, \phi \rangle = \sum_j \langle a, \phi_j \rangle \langle b, \phi'_j \rangle \tag{39}$$

with

$$\Delta(\phi) = \sum_j \phi_j \phi'_j. \tag{40}$$

Here ϕ and a, b are elements of a Hopf algebra and its dual. Inserting in (39) $a = \theta_\alpha$, $b = \lambda$ and $\phi = Q_\alpha$ and using (20), (38) we get

$$\lambda\theta_\alpha = q\theta_\alpha\lambda. \tag{41}$$

Taking $a = x_{nm}$, $b = \lambda$ and $\phi = X_{nm}$ we conclude that elements x_{nm} commute with λ. The choice $a = \lambda^2$, $b = \lambda$ and $\phi = K$ implies $\langle \lambda^3, K \rangle = 1$. Since λ^3 cannot be proportional to the diagonal elements x_{nn} ($\langle \lambda^3, X_{nn} \rangle = 0$) we have

$$\lambda^3 = 1. \tag{42}$$

The above condition can be shown to imply the comultiplication

$$\Delta(\lambda) = \lambda \otimes \lambda. \tag{43}$$

To make (41) and (43) compatible we have to define

$$\Delta(\theta_\alpha) = \sum_{\beta=1}^{N} \theta_\beta \otimes d_{\beta\alpha} + \lambda \otimes \theta_\beta + \cdots \tag{44}$$

where $d = \{d_{\alpha\beta}\}$ is an N-dimensional representation of a Lie group G under consideration and $\cdots$ denote the combination of terms consisting of 4, 7, 10 and so on θ_α generators. Using (44), (38) and (39) after some algebra we get

$$\{\theta_\alpha, \theta_\beta, \theta_\gamma\} = 0. \tag{45}$$

In a similar way the commutativity of x_{nm} with λ and (43) implies

$$\Delta(x_{nm}) = \sum_k x_{nk} \otimes x_{km} + \cdots \tag{46}$$

where $\cdots$ denotes the combination of terms consisting of 3, 6, 9 and so on θ_α generators. Using (46) and (39) we conclude that elements x_{nm} commute with θ_α.

Let us denote the algebra generated by θ_α, $\alpha = 1, \ldots, N$ and λ satisfying (41), (42) and (45) by Λ_3^N and the direct product algebra $A(G) \times \Lambda_3^N$ by $A_3^N(G)$. This algebra is in nondegenerate duality with a Hopf algebra $U_3^N(g)$. We call $A_3^N(G)$ the fractional supergroup. Using the properties

$$\varepsilon(a) = \langle a, 1\rangle \tag{47}$$

and

$$\langle S(a), \phi\rangle = \langle a, S(\phi)\rangle \tag{48}$$

of the duality relations we get the counite operation

$$\varepsilon(x_{nm}) = \delta_{nm} \qquad \varepsilon(\theta_\alpha) = 0 \qquad \varepsilon(\lambda) = 1 \tag{49}$$

and the antipode

$$S(\lambda) = \lambda^2. \tag{50}$$

Using the properties of duality relations and axioms of a Hopf algebra one can derive unknown terms in (44) and (46) and antipodes $S(x_{nm})$, $S(\theta_\alpha)$. These calculations depends on structure constants c^i_{jk}, $a^j_{\alpha\beta}$ and $b^j_{\alpha\beta\gamma}$. We demonstrate this construction on the explicit examples which will be given later.

Before closing this section we define S_n-graded Lie algebras and groups. This can be done in the same way as the S_3 case. For this one has to use the S_n invariant form

$$\{Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_n}\} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n \in S_n} Q_{\alpha_1} Q_{\alpha_2} \cdots Q_{\alpha_n} \tag{51}$$

where summation runs over all permutations of S_n. Instead of (16) and (18) we then have

$$\{Q_{\alpha_1}, Q_{\alpha_2}, \ldots, Q_{\alpha_n}\} = b^j_{\alpha_1\alpha_2,\ldots,\alpha_n} X_j \tag{52}$$

and

$$K Q_\alpha = q Q_\alpha K \qquad q^n = 1 \qquad K^n = 1 \tag{53}$$

such that

$$\sum_{\sigma=1}^{N} \sum_{(\alpha_1,\ldots,\alpha_n)\in Z_n} a^k_{\alpha_1\sigma} b^i_{\sigma_2\ldots\alpha_n} = \sum_{j=1}^{\dim g} c^i_{jk} b^j_{\alpha_1\alpha_2\ldots\alpha_n} \tag{54}$$

and

$$\sum_{k=1}^{\dim(g)} \sum_{(\alpha_1,\ldots,\alpha_{n+1})\in Z_{n+1}} b^k_{\alpha_1\ldots\alpha_n} a^k_{\alpha_{n+1}\tau} = 0. \tag{55}$$

The multiplication and counite in $U_n^N(g)$ are similar to that in $U_2^N(g)$ or $U_3^N(g)$ while the antipode is given by

$$S(Q_\alpha) = -K^{n-1} Q_\alpha \qquad S(K) = K^{n-1}. \tag{56}$$

The fractional supergroup is the algebra $A_n^N(G) = A(G) \times \Lambda_n^N$ where Λ_n^N is the algebra generated by θ_α, $\alpha = 1, \ldots, N$, λ with relations

$$\{\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_n}\} = 0 \qquad \alpha_k \in 1, 2, \ldots, N \tag{57}$$

and

$$\lambda\theta_\alpha = q\theta_\alpha\lambda \qquad \lambda^n = 1. \tag{58}$$

The co-algebra operations and the antipode in $A_n^N(G)$ depend on the structure constants c_{ij}^k, $a_{\alpha\beta}^j$ and $b_{\alpha_1\alpha_2,\ldots,\alpha_n}^j$. As an example let us consider the fractional superalgebra

$$Q^n = P \qquad [X, Q] = 0. \tag{59}$$

Since we have only one superelement Q the S_n invariant form is equal up to the multiple to Q^n. The corresponding fractional group is generated by θ, z and λ such that

$$\theta^n = 0 \qquad \lambda^n = 1 \qquad \lambda\theta = q^n\theta\lambda \qquad q^n = 1 \tag{60}$$

with z being commutative with θ and λ. The duality relations are

$$\langle Q, \theta\rangle = 1 \qquad \langle X, z\rangle = 1 \qquad \langle K, \lambda\rangle = q. \tag{61}$$

Using properties of duality relations we arrive at the following coalgebra structure:

$$\Delta(\theta) = \theta \otimes 1 + \lambda \otimes \theta \tag{62}$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{k=1}^{n-1} \frac{\lambda^{n-k}\theta^k \otimes \theta^{n-k}}{(q:q)_k(q:q)_{n-k}} \tag{63}$$

$$\varepsilon(\theta) = 0 \qquad \varepsilon(\lambda) = 1 \tag{64}$$

and

$$S(\theta) = -\lambda^{n-1}\theta \qquad S(\lambda) = \lambda^{n-1} \tag{65}$$

where

$$(q;q)_k = \prod_{j=1}^{k}(1 - q^j). \tag{66}$$

4. S_3 graded superalgebras $sl(2)$

From the commutation relations

$$[X_1, X_2] = X_3 \qquad [X_3, X_1] = 2X_1 \qquad [X_3, X_2] = -2X_2 \tag{67}$$

for the algebra $sl(2)$ we read

$$c_{12}^3 = 1 \qquad c_{31}^1 = 2 \qquad c_{32}^2 = -2. \tag{68}$$

For given N the matrix $a^j = \{a_{\alpha\beta}^j\}$ due to (32) is an arbitrary N-dimensional representation of $sl(2)$. The solution of (35) and (36) for $b_{\alpha\beta\gamma}^j$ is fully determined by this representation. Since $b_{\alpha\beta\gamma}^j$ is symmetric in α, β and γ through (16) the number of unknown coefficients for the $sl(2)$ case is $N(N+1)(N+2)/2$. On the other hand, equation (35) which is symmetric in $\alpha,\ \beta,\ \gamma$ gives $3N(N+1)(N+2)/2$ equations and equation (36) which is symmetric in $\alpha,\ \beta,\ \gamma,\ \sigma$ gives $N^2(N+1)(N+2)(N+3)/24$ equations. Although the system seems overdetermined there are solutions of which some will be given below. We consider $N = 1, 2$ and 3 fractional supergeneralizations of $sl(2)$ at $n = 3$, that is $q = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$.

4.1. $N = 1$ fractional super-sl(2)

We have one supergenerator Q_1 which can transform as a scalar only. Therefore $a^j_{11} = 0$. Inserting it in the relations (35) and (36) we get $b^j_{111} = 0$. These structure constants imply that the fractional superalgebra $U^1_3(sl(2))$ is the direct product of the universal enveloping algebra $U(sl(2))$ and the Hopf algebra generated by Q_1 and K satisfying the relations

$$KQ_1 = qQ_1K \qquad Q_1^3 = 0 \qquad K^3 = 1 \tag{69}$$

and the co-algebra operations (20) and (21). The fractional supergroup $A^3_1(SL(2))$ is the direct product of the Hopf algebras $A(SL(2))$ and Λ^1_3. Recall that the Hopf algebra structure of polynomial algebra $A(SL(2))$ is given by

$$\Delta(x_{nm}) = \sum_{k=1}^{2} x_{nk} \otimes x_{km} \tag{70}$$

and

$$S(x_{11}) = x_{22} \qquad S(x_{22}) = x_{11} \qquad S(x_{12}) = -x_{12} \qquad S(x_{21}) = -x_{21} \tag{71}$$

where two by two matrix $x = \{x_{nm}\}$ representing $SL(2)$ has determinant 1. The Hopf algebra structure of the algebra Λ^1_3 is given by the following formulae:

$$\Delta(\theta_1) = \theta_1 \otimes 1 + \lambda \otimes \theta_1 \qquad \Delta(\lambda) = \lambda \otimes \lambda \tag{72}$$

$$S(\theta_1) = -\lambda^2\theta_1 \qquad S(\lambda) = \lambda^2. \tag{73}$$

4.2. $N = 2$ fractional super-sl(2)

For $N = 2$ we have two possibilities. We can either require generators Q_1, Q_2 to transform as scalars or as spinors.

(i) In the former case we have $a^j_{\alpha,\beta} = 0$. From the relations (35) and (36) we get $b^j_{\alpha\beta\gamma} = 0$. The obtained structure constants imply that the fractional superalgebra $U^2_3(sl(2))$ is the direct product of the universal enveloping algebra $U(sl(2))$ and the Hopf algebra generated by Q_1, Q_2 and K satisfying the relations

$$KQ_\alpha = qQ_\alpha K \qquad \{Q_\alpha, Q_\beta, Q_\gamma\} = 0 \qquad K^3 = 1 \tag{74}$$

and the co-algebra operations (20) and (21). The fractional supergroup $A^3_2(SL(2))$ is the direct product of the Hopf algebras $A(SL(2))$ and Λ^2_3. The Hopf algebra structure of Λ^2_3 is given by the following formulae:

$$\Delta(\theta_\alpha) = \theta_\alpha \otimes 1 + \lambda \otimes \theta_\alpha \qquad \Delta(\lambda) = \lambda \otimes \lambda \tag{75}$$

$$S(\theta_\alpha) = -\lambda^2\theta_\alpha \qquad S(\lambda) = \lambda^2. \tag{76}$$

(ii) Let us now assume that Q_1 and Q_2 transforms as spinors under the action of $sl(2)$. We have

$$a^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{77}$$

Equation (36) gives ten equations for 12 unknowns:

$$b^1_{111} = b^1_{112} = b^2_{122} = b^2_{222} = b^3_{111} = b^3_{222} = 0 \tag{78}$$

$$b^1_{122} = -\tfrac{1}{3}b^2_{111} = b^3_{112} \qquad b^1_{222} = -3b^2_{112} = 3b^3_{122}. \tag{79}$$

Substituting these into (35) one finds that the only solution is $b^j_{\alpha\beta\gamma} = 0$. Thus we obtained the following fractional superalgebra:

$$\{Q_\alpha, Q_\beta, Q_\gamma\} = 0 \tag{80}$$

$$[Q_1, X_1] = Q_2 \qquad [Q_2, X_2] = Q_1 \qquad [Q_1, X_3] = Q_1 \qquad [Q_2, X_3] = -Q_2. \tag{81}$$

Using the general construction given in the previous section one can define the fractional supergroup $A_3^1(SL(2))$ corresponding to the above fractional superalgebra. $A_3^1(SL(2))$ is the algebra generated by elements x_{nm}, θ_n, $n, m = 1, 2$ and λ satisfying (57), (58) and $\det(x_{nm}) = 1$. The co-algebra operations and antipode can be shown to be given by (70), (71) and

$$\Delta(\theta_1) = \theta_2 \otimes x_{21} + \theta_1 \otimes x_{11} + \lambda \otimes \theta_1 \tag{82}$$

$$\Delta(\theta_2) = \theta_2 \otimes x_{22} + \theta_1 \otimes x_{12} + \lambda \otimes \theta_2 \tag{83}$$

and

$$S(\lambda) = \lambda^2 \qquad S(\theta_1) = \lambda^2(x_{21}\theta_2 - x_{22}\theta_1) \qquad S(\theta_2) = \lambda^2(x_{12}\theta_1 - x_{11}\theta_2). \tag{84}$$

The duality relations are given by the formulae

$$\langle X_3, x_{nn}\rangle = (-)^{n+1} \qquad \langle X_1, x_{12}\rangle = 1 \qquad \langle K, x_{nm}\rangle = \delta_{nm} \tag{85}$$

$$\langle X_2, x_{21}\rangle = 1 \qquad \langle Q_\alpha, \theta_\beta\rangle = \delta_{\alpha\beta} \qquad \langle K, \lambda\rangle = q. \tag{86}$$

4.3. $N = 3$ *fractional super-*$sl(2)$

We have three different superalgebras depending on the choice of a^j.

(i) Take $a^j_{\alpha\beta} = 0$. The relations (35) and (36) imply $b^j_{\alpha\beta\gamma=0}$. This case is similar with (i) of 4.2.

(ii) Take the vector representation

$$a^1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \qquad a^2 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{87}$$

The substitution of

$$a^1_{21} = a^1_{32} = a^2_{12} = a^2_{23} = \sqrt{2} \qquad a^3_{11} = -2 \qquad a^3_{33} = 2 \tag{88}$$

into (36) gives

$$\begin{aligned}
b^1_{111} &= 3\sqrt{2}b^3_{112} = -3b^2_{113} \\
b^1_{112} &= \sqrt{2}b^3_{122} = -2b^2_{123} \\
b^1_{122} &= \frac{\sqrt{2}}{3}b^3_{222} = -b^2_{223} \\
b^1_{113} &= 2\sqrt{2}b^3_{123} = -b^2_{133} \\
b^1_{133} &= \sqrt{2}b^3_{233} = -\tfrac{1}{3}b^2_{333} \\
b^1_{123} &= \frac{\sqrt{2}}{2}b^3_{223} = -\frac{1}{2}b^2_{233}
\end{aligned} \tag{89}$$

and the remaining 12 parameters $b^j_{\alpha\beta\gamma}$ are zero. The substitution of (89) into (35) gives

$$b^1_{113} = -2b^1_{122} = -b^2_{133} = 2b^2_{223} = 2\sqrt{2}b^3_{123} = -\frac{2\sqrt{2}}{3}b^3_{222} \tag{90}$$

and all other $b^j_{\alpha\beta\gamma}$ are zero. Hence we have a unique extension for the vector representation of $sl(2)$. Equations satisfied by $b^j_{\alpha\beta\gamma}$, namely (35) and (36), are invariant under rescaling $b^j_{\alpha\beta\gamma} \to kb^j_{\alpha\beta\gamma}$ where k is any nonzero constant. The choice of this nonzero constant results only in a rescaling of the generators Q_α. Just for the sake of simplicity we choose $Q_2^3 = X_3$, i.e. $b^3_{222} = 6$. Then the fractional supersymmetric extension of $sl(2)$ reads

$$[Q_1, X_2] = \sqrt{2}Q_2 \qquad [Q_1, X_3] = -2Q_1 \qquad [Q_2, X_1] = \sqrt{2}Q_1 \tag{91}$$

$$[Q_2, X_2] = \sqrt{2}Q_3 \qquad [Q_3, X_1] = \sqrt{2}Q_2 \qquad [Q_3, X_3] = 2Q_3 \tag{92}$$

and

$$\{Q_1, Q_1, Q_3\} = -4\sqrt{2}X_1 \qquad \{Q_1, Q_2, Q_2\} = 2\sqrt{2}X_1 \qquad \{Q_1, Q_2, Q_3\} = -2X_3 \quad (93)$$

$$\{Q_1, Q_3, Q_3\} = -4\sqrt{2}X_2 \qquad \{Q_2, Q_2, Q_2\} = 6X_3 \qquad \{Q_2, Q_2, Q_3\} = -2\sqrt{2}X_2. \quad (94)$$

Note also that all $b^j_{\alpha\beta\gamma} = 0$ is always a solution of (35) and (36).

(iii) Assume that two of the fractional supergenerators Q_1, Q_2 and Q_3 transform as spinors and the remaining one transforms as a scalar, that is

$$a^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (95)$$

The conditions (35) and (36) imply

$$b^1_{223} = -b^2_{111} = 2b^3_{123} \quad (96)$$

with all other structure coefficients $b^j_{\alpha\beta\gamma}$ being zero. Choosing $b^1_{223} = 1$ we get the fractional superalgebra given by (81) and

$$\{Q_1, Q_1, Q_3\} = -X_2 \qquad \{Q_2, Q_2, Q_3\} = X_1 \qquad \{Q_1, Q_2, Q_3\} = \tfrac{1}{2}X_3 \quad (97)$$

with all other relations being zero.

Before closing this section we discuss realization of fractional superalgebras by 'differential operators' in some linear spaces. Recall that for realization of superalgebras one uses superderivatives which act on superspaces. Let $F(M)$ be an algebra of functions on a manifold M. For fixed grading n and the number N of 'grassmannian' variables a fractional superspace is defined to be the direct product algebra $F(M) \times \Lambda^N_n$. We define fractional derivatives D_{θ_α} by the formulae

$$D_{\theta_\alpha}\theta_\beta = \delta_{\alpha\beta} \qquad D_{\theta_\alpha}(ab) = D_{\theta_\alpha}(a)b + k(a)D_{\theta_\alpha}(b) \quad (98)$$

where $a,\ b \in \Lambda^N_n$ and

$$k(\theta_\alpha) = q\theta_\alpha \qquad k(ab) = k(a)k(b). \quad (99)$$

Note that $D_{\theta_\alpha}(f) = 0$ and $k(f) = f$ if $f \in F(M)$. One can verify that these derivatives satisfy the relations

$$\sum_{\alpha_1\ldots\alpha_n\in S_n} D_{\theta_{\alpha_1}} \cdots D_{\theta_{\alpha_n}} = 0. \quad (100)$$

Using the fractional derivatives and superspaces defined above one can construct a realization of a fractional superalgebras. For example, the formulae

$$X_1 = -z^2\frac{\mathrm{d}}{\mathrm{d}z} - zL \qquad X_2 = \frac{\mathrm{d}}{\mathrm{d}z} \qquad X_3 = 2z\frac{\mathrm{d}}{\mathrm{d}z} + L \quad (101)$$

$$Q_1 = D_\theta \qquad Q_2 = -zD_\theta \qquad Q_3 = \frac{q}{2}\theta^2\frac{\mathrm{d}}{\mathrm{d}z} \qquad K = q^L \quad (102)$$

where $q = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ and

$$L = -q(2\theta^2 D^2_\theta + D_\theta\theta^2 D_\theta) \quad (103)$$

define representation of the fractional algebra (iii) in the linear space $A(C) \times \Lambda^1_3$, where $A(C)$ is the algebra of polynomials of the complex variable z. Indeed, using

$$L\theta^k = k\theta^k \quad (104)$$

and

$$\theta^2 D^2_\theta + D^2_\theta\theta^2 + D_\theta\theta^2 D_\theta = -q^2 \quad (105)$$

one can easily verify the relations (67), (81) and (97).

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